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COMMENT

A note on Coniglio's lemma

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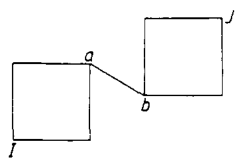
Abstract. It is proved that Coniglio's lemma holds for any pair of points in any graph in the low-density series expansion of the cut-edge weighted pair connectedness.

In a recent letter, Hong and Stanley (1983) report that Coniglio's lemma (Coniglio 1981, 1982) apparently holds 'term by term' for each graph in the low-density series expansion of the cut-edge weighted pair connectedness. We show that this is only to be expected from the observations below, which establish Coniglio's result as an elementary property of the pair connectedness of any pair of points on any graph.

Let us take an explicit example. Consider the graph



and suppose the bonds conduct with probability p. Using elementary combinatorics (Riordan 1958), the expectation that I and J are bond connected (the pair connectedness) can be written down by inspection. We find:

$$\langle IJ \rangle = 4p^5 - 4p^7 + p^9. \tag{1}$$

Likewise the expectation that I and J are bond connected, *weighted* by the number of cut edges in the spanning cluster (red bonds in the terminology of Coniglio), is found to be:

$$\langle RIJ \rangle = 20p^5 - 28p^7 + 9p^9. \tag{2}$$

In this example

$$\langle RIJ \rangle = p\, \mathrm{d}(\langle IJ \rangle)/\mathrm{d}p. \tag{3}$$

This is essentially the property noticed by Hong and Stanley. We now prove the relationship (3) in a way which is perfectly general and applies to any pair of points on any graph.

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Consider any edge E, which conducts with probability ρ, and let all the other edges conduct with probability p. Then the pair connectedness between any two points I, J can be written

$$\langle IJ\rangle = \langle IJ\rangle_E + \langle IJ\rangle_{\bar{E}} \tag{4}$$

where the first contribution results from choices of spanning cluster for which E is a cut edge and the second for which it is not. In our specific example above, if E is any edge except the edge ab:

$$\langle IJ\rangle_E = (2p^4 - 3p^6 + p^8)\rho \qquad \text{and} \qquad \langle IJ\rangle_{\bar{E}} = (2p^5 - p^7) \tag{5}$$

and if E is the edge ab:

$$\langle IJ\rangle_E = (4p^4 - 4p^6 + p^8)\rho \qquad \text{and} \qquad \langle IJ\rangle_{\bar{E}} = 0. \tag{6}$$

But we must always have, for any graph:

$$\langle IJ\rangle_E = f(p)\rho \tag{7}$$

$$\langle IJ\rangle_{\bar{E}} = g(p) \tag{8}$$

where f and g are polynomials in p uniquely determined by the combinatorics of the problem. (The first contribution must necessarily vanish with ρ; the second contribution must be independent of ρ).

In general we can therefore write

$$\langle IJ\rangle = f(p)\rho + g(p) \tag{9}$$

and

$$\rho\, \partial(\langle IJ\rangle)/\partial\rho = f(p)\rho. \tag{10}$$

But the right-hand side of (10) is just the contribution of E to $\langle RIJ\rangle$ of every choice for which E is a cut edge. Applying this argument to every edge in turn establishes (3), and, therefore, Coniglio's lemma, for any graph and for any pair of points.

The same argument (*mutatis mutandis*) applied to the site problem establishes the corresponding result for 'red' sites.

References

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